International Journal of Theoretical Physics, Vol. 33, No. 9, 1994

Intrinsic Algebraic Characterization of Space-Time Structure

Ulrich Bannier[1]

Received September 10, 1993

Starting from a partially ordered set of C^*-algebras $\mathcal{R}_i$ representing algebras of observables of physical subsystems, we derive a topological Hausdorff space $\mathcal{M}$ as a candidate for some generalized "space-time" with the help of which one can define a net $O \to \mathcal{A}(O)$, $O \subseteq \mathcal{M}$, of algebras. This opens a way to define a physical theory without an underlying metaphysical manifold, an aspect which may be relevant for the unification of general relativity and quantum field theory.

1. INTRODUCTION

The following discussion will be in the framework of local quantum physics (Haag and Kastler, 1964; Haag, 1992). Let $\mathbb{M}$ be Minkowski space. To every region (open set with compact closure) $O \in \mathbb{M}$ there corresponds a C^*-algebra $\mathcal{A}(O)$. The smallest C^*-algebra $\mathcal{A}$ containing all the $\mathcal{A}(O)$ is called the *algebra of quasilocal observables*. The fundamental insight of algebraic quantum field theory is that the entire physical information of a theory is encoded in the assignment $O \to \mathcal{A}(O)$, the *net of local algebras*, which is assumed to fulfill the following axioms, which we list for later reference:

A1. *Isotony*: $O_1 \subseteq O_2 \subseteq \mathbb{M} \Rightarrow \mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.

A2. *Einstein causality*: If two regions O_1, O_2 lie spacelike to each other, then the elements of $\mathcal{A}(O_1)$ commute with those of $\mathcal{A}(O_2)$, i.e., $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.

A3. *Primitive causality*: If the region O_2 is in the causal completion (see below) O_1'' of the region O_1, then $\mathcal{A}(O_2) \subseteq \mathcal{A}(O_1)$.

A4. *Poincaré invariance*: The orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ is represented by automorphisms $\alpha_{(a,\Lambda)}$, $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, acting on $\mathcal{A}$ by $\alpha_{(a,\Lambda)}(\mathcal{A}(O)) = \mathcal{A}(a + \Lambda O)$. Here a is a translation vector and Λ a Lorentz transformation.

[1]Fachhochschule Hamburg, Fachbereich MCh, D-20099 Hamburg, Germany.

0020-7748/94/0900-1797$07.00/0

Axiom A2 expresses the usual causal structure of $\mathbb{M}$. If O is a subset of $\mathbb{M}$, the *causal complement* O' of O is the largest set of $\mathbb{M}$ whose points lie spacelike to all points of O. We call $(O')' = O''$ the *causal completion* of O. We call O *causally complete* if $O'' = O$. Axiom A3 stipulates the existence of a dynamical law respecting the causal structure of $\mathbb{M}$. It corresponds to the hyperbolic propagation character of fields (Haag, 1992).

The algebra $\mathscr{A}$ is assumed to be the mathematical image of a *physical system* and the $\mathscr{A}(O)$ that of *physical subsystems* (see also Section 2). Self-adjoint elements of these algebras $\mathscr{A}(O)$ are assumed to detect events within the regions O and are considered as *observables*. Generally, we regard each $A \in \mathscr{A}$ as an *operation* changing the states of a system. States are represented by positive linear functionals with norm 1 on $\mathscr{A}$. In this algebraic scheme space-time is assumed *a priori* to be a four-dimensional manifold. Although this metaphysical view seems to be even possible if this manifold is some curved space-time of a fixed gravitational background (e.g., Dimock, 1980), one strongly feels that one has to analyze space-time more critically with respect to the unification of general relativity and quantum theory.

"Leibniz contended that space and time are only systems of relations. Although both physicists and philosophers tended more and more to take Leibniz's view rather than Newton's, the technique of mathematical physics continued to be Newtonian" (Russell, 1976, p. 295). "When we deny Newton's theory of absolute space, while continuing to use what we call *points* in mathematical physics, our procedure is only justified if there is a structural definition of *point* and (in theory) of particular points" (Russell, 1976, p. 296), i.e., to show "the elements and the relations constituting the structure" (Russell, 1976, p. 194).

We are, in the sense of Bohr, only able to define an experiment [and thus each $A \in \mathscr{A}(O)$] in a classical manner with our common language and concepts of space and time. Hence, Newtonian space-time of our daily life is inherent in each $A \in \mathscr{A}(O)$. But we have learned, especially from relativity theory, that although experiments are defined in a purely Newtonian way, the theoretical space-time concept necessary to describe the outcomes of experiments is non-Newtonian.

To get a structural definition of space-time in quantum field theory, it should be definable by relations between devices of operations or measurements. Actual relations like causal relations between space-time points may depend on actual states of the physical system. If $\mathscr{A}$ or $\mathscr{A}(O)$ is a well-defined set representing devices for operations or measurements, then all of their physical relations, including space-time, should be built into $\mathscr{A}$ or $\mathscr{A}(O)$. Therefore, space-time should be extractable from $\mathscr{A}$ or the $\mathscr{A}(O)$.

In the following we present a method for extracting a topological Hausdorff space $\mathcal{M}$, which may represent space-time, from a C^*-algebra $\mathcal{A}$. For this method the only relation we need to hold is a (strict) partial order relation between special subalgebras of $\mathcal{A}$. Given an algebraic structural definition of space-time $\mathcal{M}$, then a net $O \to \mathcal{A}(O)$ is a purely algebraic concept with all terms having a well-defined physical interpretation.

2. NETS OF ALGEBRAS BASED ON AN INTRINSICALLY DEFINED SPACE

The measurable properties of a physical system depend on the set of measurement devices which are available and which are sensitive for that system. One can say (in a positivistic view) that this set is the empirical representative of a physical system. Therefore, one may regard the quasilocal algebra $\mathcal{A}$ as a *physical system* and the local algebras $\mathcal{A}(O)$ as *physical subsystems.*

The idea now is that it is not the assignment $O \to \mathcal{A}(O)$, O a subset of some underlying space $\mathcal{M}$, that is fundamental, but the identification of physical subsystems and their relations to each other and that $\mathcal{M}$ and the net $O \to \mathcal{A}(O)$ are derived concepts. Let us assume that we have identified a set $\mathbf{E}$ of C^*-subalgebras of a C^*-algebra $\mathcal{A}$ which represent such physical subsystems. (For the derivation of $\mathcal{M}$ that follows it is not necessary for the elements of $\mathbf{E}$ to be algebras, but let us stay in the algebraic picture which has been so successful until now, although finally quantum gravity might demand that we abolish the algebraic structure.) Let $\mathbf{E}$ be partially ordered by a relation $\ll$. We leave it open whether $\ll$ is reflexive or antireflexive. Elements of $\mathbf{E}$ are denoted by $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_i, \ldots$. The expression $\mathcal{R}_1 \ll \mathcal{R}_2$ means that $\mathcal{R}_1$ is a subsystem which is "essentially smaller" than $\mathcal{R}_2$. (The meaning of "essentially smaller" will become clearer in the example of Section 3.)

Definition 2.1. (a) A set $\mathbf{N} \subseteq \mathbf{E}$ is called *overlapping* if each finite subset $\mathbf{I}$ of $\mathbf{N}$ has a lower bound in $\mathbf{E}$, i.e., $\exists \mathcal{R}_1 \in \mathbf{E}$: $\mathcal{R}_1 \ll \mathcal{R}$ for all $\mathcal{R} \in \mathbf{I}$.

(b) $\mathbf{N}$ is called *shrinking* if it is overlapping and $\mathcal{R} \in \mathbf{N} \Rightarrow \exists \mathcal{R}_1 \in \mathbf{N}$: $\mathcal{R}_1 \ll \mathcal{R}$.

(c) A shrinking set $\mathbf{N} \subseteq \mathbf{E}$ is called *maximal* if it is not contained in any shrinking set $\mathbf{N}_1 \neq \mathbf{N}$.

(d) The set of all maximal shrinking sets of $\mathbf{E}$ is denoted by $\mathcal{M}$ and elements of $\mathcal{M}$, also called points, by lowercase letters $x, x_1, y, \ldots$. [Compare a related definition of points of a linear space with the help of sets of overlapping convex sets by Russell (1976), p. 299.]

If $\preccurlyeq$ is reflexive, then all sets $\{\mathscr{R}\}$ consisting of a single element $\mathscr{R} \in \mathbf{E}$ are shrinking because $\mathscr{R} \preccurlyeq \mathscr{R}$. The existence of maximal shrinking sets follows by Zorn's Lemma and in this case $\mathscr{M}$ is not empty. If $\preccurlyeq$ is antireflexive, then the existence of shrinking sets is an additional property of $\mathbf{E}$.

Definition 2.2. We call the pair $(\mathbf{E}, \preccurlyeq)$, or simply $\mathbf{E}$, the *base* of a net if:

(a) $\mathscr{R}_1 \preccurlyeq \mathscr{R}_2 \Rightarrow \mathscr{R}_1 \subseteq \mathscr{R}_2$.
(b) $\mathscr{R} \in \mathbf{E} \Rightarrow \exists \mathscr{R}_1 \in \mathbf{E}: \mathscr{R}_1 \preccurlyeq \mathscr{R}$.

Property (a) guarantees the isotony property of the net of algebras to be derived and (b) the existence of shrinking sets and, therefore, $\mathscr{M} \neq \varnothing$. In the following we always regard $\mathbf{E}$ as a base of a net. Furthermore, let $\mathscr{A}$ be the smallest C^*-algebra containing all elements of $\mathbf{E}$.

Definition 2.3. For a finite subset $\mathbf{I} \subseteq \mathbf{E}$ we define $O_{\mathbf{I}} = \{x \in \mathscr{M} \mid \mathbf{I} \subseteq x\}$. If $\mathbf{I} = \{\mathscr{R}\}$, we simply write $O_{\mathscr{R}}$.

If $\mathbf{I} \subseteq \mathbf{E}$ is finite and not overlapping in $\mathbf{E}$, then $O_{\mathbf{I}} = \varnothing$, because in this case the set $\mathbf{I}$ cannot be contained in any maximal shrinking set $x \in \mathscr{M}$.

Proposition 2.4. The sets $O_{\mathbf{I}}$, $\mathbf{I} \subseteq \mathbf{E}$ finite, form a base of a topology $\mathscr{T}$ for $\mathscr{M}$, i.e.:

(a) $\mathscr{M} = \bigcup_{\mathbf{I}} O_{\mathbf{I}}$.
(b) $\bigcap_{i=1}^n O_{\mathbf{I}_i} = O_{\mathbf{J}}$ with $\mathbf{J} = \bigcup_{i=1}^n \mathbf{I}_i$.

In particular, $\mathscr{T}$ is a Hausdorff topology for $\mathscr{M}$.

Proof. (a) For every $x \in \mathscr{M}$ there exists a finite, overlapping set $\mathbf{I} \subseteq x \subseteq \mathbf{E}$, and $x \in \mathscr{M}$ is contained in $O_{\mathbf{I}}$.

(b) We have

$$\bigcap_{i=1}^n O_{\mathbf{I}_i} = \{x \in \mathscr{M} \mid \mathbf{I}_i \subseteq x, \forall i = 1, \ldots, n\} = \left\{x \in \mathscr{M} \,\middle|\, \bigcup_{i=1}^n \mathbf{I}_i = \mathbf{J} \subseteq x\right\} = O_{\mathbf{J}}$$

where $\mathbf{J}$ is a finite set.

It remains to show that $\mathscr{T}$ is a Hausdorff topology. Let $x, y \in \mathscr{M}$, $x \neq y$. If $x \cup y$ is overlapping, then $x \cup y$ is shrinking. By $x \subset (x \cup y)$ we get a contradiction because x already is a maximal shrinking set. Therefore, there exists a finite set $\mathbf{I} \subseteq x \cup y$ which is not overlapping and $\mathbf{I} \cap x \neq \varnothing$, $\mathbf{I} \cap y \neq \varnothing$. As $\mathbf{I} \cap x$ and $\mathbf{I} \cap y$ are finite sets, we conclude $x \in O_{\mathbf{I} \cap x}$, $y \in O_{\mathbf{I} \cap y}$, and $O_{\mathbf{I} \cap x} \cap O_{\mathbf{I} \cap y} = O_{\mathbf{I}} = \varnothing$. ■

If $\mathscr{M}$ has been derived with help of a base of a net $(\mathbf{E}, \preccurlyeq)$, it is natural to associate with open sets $O \subseteq \mathscr{M}$ the C^*-algebras

$$\mathscr{A}(O) = \overline{\bigcup_{\substack{\mathscr{R} \in \mathbf{E} \\ O_{\mathscr{R}} \subseteq O}} \mathscr{R}} \tag{1}$$

(algebraic closure in the C^*-norm), where $O_{\mathscr{R}} = \{x \in \mathscr{M} | \mathscr{R} \in x\}$, which we again want to interpret as physical subsystems. There might be reasons to regard also the algebras of points

$$\mathscr{A}(x) = \bigcap_{x \in O} \mathscr{A}(O)$$

as (idealized) subsystems. By the definition (1) we get a net of "local" algebras $O \to \mathscr{A}(O)$, O an open set of $\mathscr{M}$. This net does not yet fulfill all axioms of algebraic field theory, but the isotony property

$$O_1 \subseteq O_2 \subseteq \mathscr{M} \;\Rightarrow\; \mathscr{A}(O_1) \subseteq \mathscr{A}(O_2)$$

is trivial.

Proposition 2.5. If there is an automorphism α on $\mathscr{A}$ with $\alpha(\mathbf{E}) = \mathbf{E}$ and which preserves the partial order relation $\preccurlyeq$, then α induces a homeomorphism κ_α of $\mathscr{M}$ by

$$x \to \kappa_\alpha(x) = \{\alpha(\mathscr{R}) | \mathscr{R} \in x\} \tag{2}$$

The inverse of κ_α is defined by $\kappa_\alpha^{-1} = \kappa_{\alpha^{-1}}$.

Proof. As α is an automorphism, $\kappa_\alpha(x)$ and $\kappa_\alpha^{-1}(x)$ are again maximal shrinking sets and κ_α and κ_α^{-1} are one-to-one. For an open set $O_{\mathscr{R}} \subseteq \mathscr{M}$ one gets

$$\kappa_\alpha^{-1}(O_{\mathscr{R}}) = O_{\alpha^{-1}(\mathscr{R})}$$

which is open for every open set $O_{\mathscr{R}}$. As the $O_{\mathscr{R}}$ form a subbase for the topology $\mathscr{T}$, κ_α and κ_α^{-1} are continuous. ■

Given a group $\mathscr{D}$ of automorphisms α of the algebra $\mathscr{A}$ with the properties of Proposition 2.5, one has by (2) a "geometric" action of this group on $\mathscr{M}$ by the κ_α and especially the following covariance property:

Proposition 2.6. We have

$$\alpha(\mathscr{A}(O)) = \mathscr{A}(\kappa_\alpha(O)), \qquad O \in \mathscr{M}$$

Proof. As κ_α is a homeomorphism, $\kappa_\alpha(O)$ is open if O is open. With $\kappa_\alpha(O_{\mathscr{R}}) = O_{\alpha(\mathscr{R})}$ one gets

$$\alpha(\mathscr{A}(O)) = \alpha\left(\overline{\bigcup_{\substack{\mathscr{R} \in \mathbf{E} \\ O_{\mathscr{R}} \subseteq O}} \mathscr{R}}\right) = \overline{\bigcup_{\substack{\mathscr{R} \in \mathbf{E} \\ O_{\alpha(\mathscr{R})} \subseteq \kappa_\alpha(O)}} \alpha(\mathscr{R})} = \mathscr{A}(\kappa_\alpha(O)). \quad ■$$

3. AN EXAMPLE

Let us consider a net $O \to \mathscr{R}(O)$ of von Neumann algebras with identity $\mathbb{1}$ which fulfills the axioms of Section 1. This net will serve us as a device to define a base of a net. $\mathscr{A}$ is the smallest C^*-algebra containing all $\mathscr{R}(O)$. For each $y \in \mathbb{M}$, $y = (y^0, \mathbf{y})$, and $r \in \mathbb{R}$ let $\mathscr{K}_{y,r}$ be the causal completion of the three-dimensional ball $\{x \in \mathbb{M} \mid |\mathbf{x} - \mathbf{y}| < r\}$ in the spacelike hyperplane at y^0, sometimes also called the *diamond* or *double cone* with center y and radius r. If the center and the radius are not important we simply write $\mathscr{K}$, $\mathscr{K}_1$, $\mathscr{K}_2, \ldots$.

We need some additional assumptions which are fulfilled, for example, for the free field. For discussions of these properties and their physical prerequisites and implications see the references.

P1. *Landau property* (Landau, 1969). If $\mathscr{K}_1, \mathscr{K}_2 \subseteq \mathbb{M}$ are open double cones, then

$$\mathscr{K}_1 \cap \mathscr{K}_2 = \varnothing \Leftrightarrow \mathscr{R}(\mathscr{K}_1) \cap \mathscr{R}(\mathscr{K}_2) = \{\lambda \mathbb{1} \mid \lambda \in \mathbb{C}\} \tag{3}$$

(Disjoint double cones do not have other operations in common than multiples of the identity $\mathbb{1}$.)

P2. *Type of the local algebras*: All $\mathscr{R}(\mathscr{K})$ of double cones $\mathscr{K}$ are hyperfinite factors of type III_1 in the classification of Connes (1982).

P3. *Split property*: If $\mathscr{K}_1$ and $\mathscr{K}_2$ are two double cones of $\mathbb{M}$, $\overline{\mathscr{K}_1}$ the closure of $\mathscr{K}_1$, then

$$\overline{\mathscr{K}_1} \subset \mathscr{K}_2 \Leftrightarrow \text{the inclusion} \quad \mathscr{R}(\mathscr{K}_1) \subset \mathscr{R}(\mathscr{K}_2) \quad \text{is split} \tag{4}$$

This means that there exists a type I factor $\mathscr{N}$ with

$$\mathscr{R}(O_1) \subset \mathscr{N} \subset \mathscr{R}(O_2)$$

Property P2 is a result of Buchholz *et al.* (1987). For the following this excludes that the algebras $\mathscr{R}(\mathscr{K})$ are trivial. Concerning the split property P3 see, for instance, Doplicher and Longo (1984) and Buchholz (1974). Property P1 helps us to prove the following lemma, which says that the assignment of (nontrivial) algebras to double cones is one-to-one.

Lemma 3.1. If $\mathscr{K}_1$, $\mathscr{K}_2$ are two double cones, then

$$\mathscr{K}_1 \neq \mathscr{K}_2 \Leftrightarrow \mathscr{R}(\mathscr{K}_1) \neq \mathscr{R}(\mathscr{K}_2)$$

Proof. Suppose that $\mathscr{K}_1$ is not included in $\mathscr{K}_2$; then there is a double cone $\mathscr{K}$ in $\mathscr{K}_1$ which does not intersect $\mathscr{K}_2$. By the Landau property (3) it follows that the subalgebra $\mathscr{R}(\mathscr{K})$ of $\mathscr{R}(\mathscr{K}_1)$ is not included in $\mathscr{R}(\mathscr{K}_2)$. ∎

Definition 3.2. The set of algebras

$$\mathbf{E}_R = \{\mathscr{R}(\mathscr{K}_{x,r}) | r > R\}$$

and the partial order relation

$$\begin{aligned}\mathscr{R}(\mathscr{K}_1) \preccurlyeq \mathscr{R}(\mathscr{K}_2) &\Leftrightarrow \overline{\mathscr{K}_1} \subset \mathscr{K}_2 \\ &\Leftrightarrow \text{the inclusion} \quad \mathscr{R}(\mathscr{K}_1) \subset \mathscr{R}(\mathscr{K}_2) \quad \text{is split} \qquad (5)\end{aligned}$$

define a base of a net $(\mathbf{E}_R, \preccurlyeq)$.

As $\mathscr{R}(\mathscr{K}_1) \preccurlyeq \mathscr{R}(\mathscr{K}_2)$ implies $\mathscr{R}(\mathscr{K}_1) \subset \mathscr{R}(\mathscr{K}_2)$ and to every $\mathscr{R}(\mathscr{K}_2)$ there exists a $\mathscr{R}(\mathscr{K}_1)$ with $\mathscr{R}(\mathscr{K}_1) \preccurlyeq \mathscr{R}(\mathscr{K}_2)$, $(\mathbf{E}_R, \preccurlyeq)$ is a base of a net in the sense of Definition 2.2. Each $(\mathbf{E}_R, \preccurlyeq)$ defines a space $\mathscr{M}_R$.

Note that (5) reduces the partial order relation $\preccurlyeq$ between algebras to a topological relation between double cones. Overlapping of a set of algebras $\mathscr{R}(\mathscr{K})$ means that finite intersections of the associated double cones contain a double cone.

The idea to use the split property to define the partial order relation is taken from Fredenhagen (1992). In Bannier (1987) we originally defined the partial order relation $\preccurlyeq$ with the help of the translations of $\mathbb{M}$ (vector group $\mathbb{R}^4$ with the usual topology), which are represented by automorphisms of $\mathscr{A}$, i.e., for each $a \in \mathbb{R}^4$ there is an automorphism α_a such that $\alpha_a(\mathscr{A}(O)) = \mathscr{A}(O + a)$. The time translations are denoted by α_t, $t \in \mathbb{R}$. One can define $\preccurlyeq$ by

$$\mathscr{R}(\mathscr{K}_1) \preccurlyeq \mathscr{R}(\mathscr{K}_2) \Leftrightarrow \begin{cases}\text{there is an open neighborhood } \mathscr{U} \\ \text{of the origin of } \mathbb{R}^4 \text{ such that} \\ \alpha_a(\mathscr{R}(\mathscr{K}_1)) \subset \mathscr{R}(\mathscr{K}_2)\ \forall a \in \mathscr{U}\end{cases}$$

or by

$$\mathscr{R}(\mathscr{K}_1) \preccurlyeq \mathscr{R}(\mathscr{K}_2) \Leftrightarrow \begin{cases}\text{there is an open neighborhood } \mathscr{U} \\ \text{of the origin of } \mathbb{R} \text{ such that} \\ \alpha_t(\mathscr{R}(\mathscr{K}_1)) \subset \mathscr{R}(\mathscr{K}_2)\ \forall t \in \mathscr{U}\end{cases}$$

In both cases one gets the same spaces $\mathscr{M}_R$ as with definition (5).

In the proof of Proposition 3.4 we shall use Helly's Theorem (Helly, 1923; Radon, 1921), although one can prove it by the finite intersection property of compact sets. Helly's Theorem shows an interesting connection between overlapping properties of convex sets and the dimension of a linear space and we give it for the convenience of the reader in the formulation of Comfort and Gordon (1961).

Theorem 3.3. (Helly's Theorem). Let D be an n-dimensional, real, normed linear space, and let $\mathscr{F}$ be a collection of compact, convex subsets of D. If every $n+1$ elements of $\mathscr{F}$ have a point in common, then some point of D lies in every element of $\mathscr{F}$.

Proposition 3.4. Each Hausdorff space $\mathscr{M}_R$ generated by $(\mathbf{E}_R, \ll)$, $R \geq 0$, is homeomorphic to $\mathbb{R}^4$.

Proof. To each $p \in \mathbb{R}^4$ we assign the set

$$x_p = \{\mathscr{R}(\mathscr{K}) \in \mathbf{E}_R | \overline{\mathscr{K}}_{p,R} \subset \mathscr{K}\} \tag{6}$$

$\overline{\mathscr{K}}_{p,R}$ is the closure of $\mathscr{K}_{p,R}$ and we define $\overline{\mathscr{K}}_{p,0} = \{p\}$ to unify the notation.

We shall first prove that the x_p are maximal shrinking sets in $\mathbf{E}_R$ and that the mapping $\mu_R: p \to x_p$ is one-to-one from $\mathbb{R}^4$ into $\mathscr{M}_R$. Each set x_p is overlapping because for each finite set of double cones $\mathscr{K}_i$ containing $\overline{\mathscr{K}}_{p,R}$ there exists a double cone $\mathscr{K}_0$ containing $\overline{\mathscr{K}}_{p,R}$ and $\mathscr{R}(\mathscr{K}_0) \ll \mathscr{R}(\mathscr{K}_i)$. As $\mathscr{R}(\mathscr{K}_0)$ itself is in x_p, we conclude that x_p is shrinking. Let us assume that x_p is not a maximal shrinking set. Then there exists an algebra $\mathscr{R}(\mathscr{K}_2) \notin x_p$ and $x_p \cup \{\mathscr{R}(\mathscr{K}_2)\}$ should be overlapping. Let $\mathscr{R}(\mathscr{K}_1) \in x_p$; then there exists an algebra $\mathscr{R}(\mathscr{K}_3) \in x_p$ with $\mathscr{R}(\mathscr{K}_3) \ll \mathscr{R}(\mathscr{K}_1)$ and $\mathscr{R}(\mathscr{K}_3) \ll \mathscr{R}(\mathscr{K}_2)$. From the last inequality it follows especially that $\mathscr{K}_3 \subset \mathscr{K}_2$. As $\mathscr{K}_3$ contains $\overline{\mathscr{K}}_{p,R}$, also $\mathscr{K}_2$ should contain $\overline{\mathscr{K}}_{p,R}$. This is a contradiction and x_p is already a maximal shrinking set in $\mathbf{E}_R$. If $p_1, p_2 \in \mathbb{R}^4$, $p_1 \neq p_2$, we can find double cones $\mathscr{K}_i$, $i = 1, 2, p_i \in \mathscr{K}_i$, which do not simultaneously contain $\overline{\mathscr{K}}_{p_1,R}$ and $\overline{\mathscr{K}}_{p_2,R}$. Therefore, we get $x_{p_1} \neq x_{p_2}$ and μ_R is one-to-one. In this sense we have $\mathbb{R}^4 \subseteq \mathscr{M}_R$.

We shall prove now that the mapping μ_R is in fact onto $\mathscr{M}_R$. For each $x \in \mathscr{M}_R$ we define

$$\mathbf{K}_x = \{\mathscr{K} | \mathscr{R}(\mathscr{K}) \in x\}$$

and for each $\mathscr{K} \in \mathbf{K}_x$

$$A_{\mathscr{K}} = \{p \in \mathbb{R}^4 | \overline{\mathscr{K}}_{p,R} \subseteq \overline{\mathscr{K}}\}$$

Each $A_{\mathscr{K}_{p,r}}$ is equal to $\overline{\mathscr{K}}_{p,r-R}$, which is a compact convex subset of $\mathbb{R}^4$. As x is overlapping, we have for each selection of $4+1$ sets $A_{\mathscr{K}_i}$, $\mathscr{K}_i \in \mathbf{K}_x$,

$$\bigcap_{i=1}^{5} A_{\mathscr{K}_i} = \{p \in \mathbb{R}^4 | \overline{\mathscr{K}}_{p,R} \subseteq \overline{\mathscr{K}}_i, i = 1, \ldots, 5\} \neq \varnothing$$

as there is a set $\mathscr{K}_0 \subset \bigcap_{i=1}^5 \mathscr{K}_i$, $\mathscr{K}_0 \in \mathbf{K}_x$, which contains at least one $\overline{\mathscr{K}}_{p,R}$. By Helly's Theorem we then get

$$\bigcap_{\mathscr{K} \in \mathbf{K}_x} A_{\mathscr{K}} = \{p \in \mathbb{R}^4 | \overline{\mathscr{K}}_{p,R} \subseteq \overline{\mathscr{K}}\} \neq \varnothing$$

This implies that there exists at least one $\overline{\mathscr{K}}_{p,R}$ contained in all $\overline{\mathscr{K}}$, $\mathscr{K} \in \mathbf{K}_x$. But then this $\overline{\mathscr{K}}_{p,R}$ is also contained in each $\mathscr{K} \in \mathbf{K}_x$ because for each

$\mathcal{K} \in \mathbf{K}_x$ there exists a $\mathcal{K}_1 \in \mathbf{K}_x$ with $\overline{\mathcal{K}_1} \subset \mathcal{K}$, as x is shrinking and $\overline{\mathcal{K}_1}$ contains $\overline{\mathcal{K}_{p,R}}$. Therefore, we get $x \subseteq x_p$. As x is already a maximal shrinking set, we conclude $x = x_p$. Every $x \in \mathcal{M}_R$ is given by (6) and we can identify $\mathbb{R}^4$ and $\mathcal{M}_R$ as sets.

It remains to show that μ_R is a homeomorphism. Let us denote the topology on $\mathcal{M}_R$ defined by Proposition 2.4 by $\mathcal{T}_R$. As the mapping μ_R is bijective, it suffices to show that the topology on $\mathbb{R}^4$ induced by μ_R is identical with the usual topology on $\mathbb{R}^4$.

Every $x \in \mathcal{M}_R$ is of the form $x_p = \mu_R(p)$ for some $p \in \mathbb{R}^4$. Let $\mathcal{R}(\mathcal{K}_i)$, $i \in \mathbf{I}$, be a finite subset of $\mathbf{E}_R$, $\mathcal{K}_i = \mathcal{K}_{q_i, r_i}$; then open sets

$$\begin{aligned} O_{\mathbf{I}} &= \{x_p \in \mathcal{M}_R \mid \mathcal{R}(\mathcal{K}_i) \in x_p,\ i \in \mathbf{I}\} \\ &= \{\mu_R(p) \in \mathcal{M}_R \mid p \in \mathbb{M},\ \overline{\mathcal{K}_{p,R}} \subset \mathcal{K}_i,\ i \in \mathbf{I}\} \end{aligned}$$

form a base of $\mathcal{T}_R$. As the sets

$$\mu_R^{-1}(O_{\mathbf{I}}) = \{p \in \mathbb{R}^4 \mid \overline{\mathcal{K}_{p,R}} \subset \mathcal{K}_i,\ i \in \mathbf{I}\} = \bigcap_{i \in \mathbf{I}} \mathcal{K}_{q_i, r_i - R}$$

which define the induced topology on $\mathbb{R}^4$, are also a base of its usual topology, μ_R is a homeomorphism. ■

This result is perhaps not surprising, but $(\mathbf{E}_R, \preccurlyeq)$ is purely algebraic. The double cones serve only for naming or indexing subalgebras of $\mathcal{A}$ and saying if a relation $\mathcal{R}(\mathcal{K}_1) \preccurlyeq \mathcal{R}(\mathcal{K}_2)$ is valid. The key to the fact that $\mathcal{M}_R$ can be viewed as a four-dimensional space is hidden in the special overlapping properties of the $\mathcal{R}(\mathcal{K})$, respectively, the $\mathcal{K}$, used in Helly's Theorem.

All $(\mathbf{E}_R, \preccurlyeq)$ define homeomorphic spaces $\mathcal{M}_R$ but different nets $O \to \mathcal{A}_R(O)$, $O \in \mathcal{M}_R$. If $R = 0$, we have the same situation as in usual algebraic quantum field theory, especially the algebras of points $\mathcal{A}(x)$ are multiples of the identity. This is no longer the case if $R > 0$. Now the algebras $\mathcal{A}(x)$ are no longer trivial.

At the end of this example, let us take the set of all hyperfinite type III_1 factors $\mathcal{R}$ contained in $\mathcal{A}$ as a base of a net $(\mathbf{E}, \preccurlyeq)$ with $\preccurlyeq$ generally defined by

$$\mathcal{R}_1 \preccurlyeq \mathcal{R}_2 \Leftrightarrow \text{the inclusion} \quad \mathcal{R}_1 \subset \mathcal{R}_2 \quad \text{is split.}$$

The $\mathbf{E}_R$ are subsets of $\mathbf{E}$ and the partial order relations on $\mathbf{E}_R$ are restrictions of that of $\mathbf{E}$. In this case we get the interesting property that all elements α of the automorphism group $\mathrm{Aut}(\mathcal{A})$ act by κ_α geometrically on the associated Hausdorff space $\mathcal{M}$, which is essentially larger than $\mathcal{M}_R$.

4. CAUSAL AND DYNAMICAL STRUCTURES

Usually space-time is modeled by a C^∞-manifold and a Lorentz metric. In the following we want to discuss the possibility of modeling space-time structures on $\mathscr{M}$, especially causal and dynamical structures, without using differential geometrical methods. In Bannier (1987, 1988) we associated with each Klein–Gordon equation

$$(\Box_{\mathbf{g}} - m^2)\Phi_{\mathbf{g}} = 0, \qquad \mathbf{g} \in \mathscr{G}$$

where $\mathbf{g}$ is an element of the set $\mathscr{G}$ of all globally hyperbolic Lorentz metrics on $\mathbb{R}^4$, a two-sided, norm-closed maximal ideal $\mathscr{I}_{\mathbf{g}}$ in a C^*-algebra $\mathscr{A}$ defined by a net $O \to \mathscr{A}(O)$, $O \subset \mathbb{R}^4$. In this net one can show that the following result holds:

Theorem 4.1. (Bannier, 1988, Theorem 2.1). (a) If two regions O_1 and O_2 of $\mathbb{R}^4$ lie spacelike to each other with respect to $\mathbf{g}$, then

$$[\mathscr{A}(O_1), \mathscr{A}(O_2)] \subseteq \mathscr{I}_{\mathbf{g}}.$$

(b) If $O_1, O_2 \subset \mathbb{R}^4$ are two regions and O_2 is in the causal completion $O_1^{\mathbf{gg}}$ (defined in analogy to O'' in the introduction) of O_1 with respect to $\mathbf{g}$, then

$$\mathscr{A}(O_2) \subseteq \mathscr{A}(O_1) + \mathscr{I}_{\mathbf{g}}$$

These statements are generalizations of Axioms A2 and A3. Motivated by this theorem, one can try to define causal and dynamical structures with the help of special ideals [which we shall define here in a slightly different way than in Bannier (1987, 1988)].

Definition 4.2. Let $(\mathbf{E}, \preccurlyeq)$ be a base of a net, $\mathscr{M}$ the associated Hausdorff space, $O \to \mathscr{A}(O)$ the associated net of algebras, $\mathscr{A}$ the quasi-local algebra generated by this net, and $\mathscr{I}$ a two-sided, norm-closed ideal of $\mathscr{A}$.

(a) Two points $p, q \in \mathscr{M}$ are defined to be *causally independent* with respect to $\mathscr{I}$ (in short: w.r.t. $\mathscr{I}$) if there are open sets $O_1, O_2 \subseteq \mathscr{M}$, $p \in O_1$, $q \in O_2$, with

$$[\mathscr{A}(O_1), \mathscr{A}(O_2)] \subseteq \mathscr{I} \tag{7}$$

$\mathscr{I}$ is called *causal* if there exist causally independent points w.r.t. $\mathscr{I}$ in $\mathscr{M}$.

(b) If $\mathscr{U}$ is any subset of $\mathscr{M}$, the *causal complement* $\mathscr{U}^{\mathscr{I}}$ of $\mathscr{U}$ w.r.t. $\mathscr{I}$ is the largest subset of $\mathscr{M}$ whose points are causally independent w.r.t. $\mathscr{I}$ to all points of $\mathscr{U}$. We call $(\mathscr{U}^{\mathscr{I}})^{\mathscr{I}} = \mathscr{U}^{\mathscr{I}\mathscr{I}}$ the *causal completion* of $\mathscr{U}$ w.r.t. $\mathscr{I}$. The set $\mathscr{U}$ is called *causally complete* w.r.t. $\mathscr{I}$ if $\mathscr{U}^{\mathscr{I}\mathscr{I}} = \mathscr{U}$.

(c) Let $\mathscr{I}$ be a causal ideal, and O_2 in the causal completion of the open set O_1 w.r.t. $\mathscr{I}$. We call $\mathscr{I}$ *hyperbolic* if

$$\mathscr{A}(O_2) = \mathscr{A}(O_1) + \mathscr{I}$$

and if for every set O_1 which is causally complete w.r.t. $\mathscr{I}$ there is an open set $O_2 \neq O_1$ with $O_2^{\mathscr{I}\mathscr{I}} = O_1$ (i.e., the structure should not be trivial).

The algebra $\mathscr{A}$ of the example in Section 3 possesses only the trivial ideal $\{0\}$. For the net $O \to \mathscr{A}_R(O)$ we recover the usual causal structure if $R = 0$. But if $R > 0$, then each $x \in \mathscr{M}_R$ has an open neighborhood $O \subset \mathscr{M}_R$ such that no point of the neighborhood is causally independent to x w.r.t. $\{0\}$. This net can perhaps be viewed as a theory with limited resolution of the measurements.

In order for a net to have well-defined causal and dynamical structures, $\mathscr{A}$ should possess hyperbolic ideals. As said in Bannier (1988), the above structures defined by ideals presumably are not the right structures for quantum gravity, but at the most an approximation, as they allow states that mix different causal structures. One can use the above definitions (a) and (c) as axioms for such a theory. For quantum gravity one would expect that causal and dynamical structures would depend directly on single states.

For all states ω which fulfill $\omega(\mathscr{I}) = 0$ we may reformulate (7) as

$$\omega([\mathscr{A}(O_1), \mathscr{A}(O_2)] = \{0\}$$

and a causal structure could have been defined by those sets of states which annihilate the same ideal. Therefore, we have reached at least a structure which depends on sets of states.

For $\mathscr{M}$ we have defined the $\mathscr{T}$-topology (compare Proposition 2.4). This is a topology which does not depend on states. The definition of causal independence w.r.t. an ideal opens the possibility to define a kind of Alexandrov topology on $\mathscr{M}$. In classical general relativity the Alexandrov topology is defined by a base consisting of sets which are intersections of the chronological past and chronological future of two points (e.g., Hawking and Ellis, 1973, p. 196). Now, for any two elements x, y of $\mathscr{M}$ which are not causally independent w.r.t. an ideal $\mathscr{I}$ let $O(x, y)^{\mathscr{I}\mathscr{I}}$ be the $\mathscr{T}$-open kernel of $\{x, y\}^{\mathscr{I}\mathscr{I}}$. If the sets $O(x, y)^{\mathscr{I}\mathscr{I}}$ are a covering of $\mathscr{M}$, then they can be regarded as a subbase for a topology of $\mathscr{M}$ which we shall denote $\mathscr{I}$-topology. In this way it is possible for a causal ideal $\mathscr{I}$ [respectively the states which fulfill $\omega(\mathscr{I}) = 0$] to define its own physical topology.

5. COMMENTS

The distinguished role classical space-time plays in every formulation of quantum field theory seems to be the source of much trouble and also the main obstacle in incorporating gravity into it. A base of a net $(\mathbf{E}, \leqslant)$ opens the possibility to define a Hausdorff space $\mathscr{M}$ and a physical theory by a net $O \to \mathscr{A}(O)$, $O \in \mathscr{M}$, completely without a presupposed metaphysical space-time. To free the definition of a theory from an underlying space might be a necessary step towards quantum gravity.

But there are still serious problems concerning this idea. Therefore, we want to pose some questions and discuss some ideas which should motivate further research.

Presumably, one can characterize a physical theory completely by a base of a net and one can view the base itself as the essential physical structure, as $\mathscr{M}$ and the net $O \to \mathscr{A}(O)$ are derived concepts. But what are the criteria that select a base of a net? One can imagine many pathological choices.

One question in this context is, by which criteria does one regard a C^*-algebra as a physical subsystem? Is it one which refers to a single algebra or is there only an implicit definition of the concept subsystem by the selection of a whole base of a net? The assumption that the algebra of a subsystem has to be a hyperfinite factor of type III_1 is not sufficient. But the type might be one selection criterion.

It seems to us that one has to select not single algebras but the whole base $(\mathbf{E}, \leqslant)$. Given a C^*-algebra $\mathscr{A}$ as in the example of Section 3, there are many bases for nets. One can regard different bases as equivalent if they define homeomorphic Hausdorff spaces $\mathscr{M}$ and isomorphic nets $O \to \mathscr{A}(O)$. There are many equivalent classes. Which class is physical? Or, are all these classes only different views onto the physical system represented by the algebra $\mathscr{A}$? Are those bases physical which define a four-dimensional space with a well-defined causal structure? Which properties of a net guarantee this?

We want to state some criteria similar to that in the example of Section 3 which may be helpful to select a base $(\mathbf{E}, \leqslant)$. Let $\mathscr{R}, \mathscr{R}_1, \mathscr{R}_2 \in \mathbf{E}$; then:

1. $\mathscr{R}_1 \cap \mathscr{R}_2 \not\subset \mathscr{Z} \Leftrightarrow \mathscr{R}_1, \mathscr{R}_2$ overlapping.
2. All $\mathscr{R}$ are hyperfinite factors of type III_1.
3. $\mathscr{R}_1 \leqslant \mathscr{R}_2 \Leftrightarrow$ the inclusion $\mathscr{R}_1 \subset \mathscr{R}_2$ is split.

$\mathscr{Z}$ is the center of $\mathscr{A}$. One can regard $\mathscr{R}_1$ and $\mathscr{R}_2$ as algebras $\mathscr{A}(O_{\mathscr{R}_1})$ and $\mathscr{A}(O_{\mathscr{R}_2})$ of the associated net on $\mathscr{M}$. If $\mathscr{R}_1 \cap \mathscr{R}_2 \not\subset \mathscr{Z}$, then nontrivial elements $A \notin \mathscr{Z}$ of $\mathscr{A}(O_{\mathscr{R}_1})$ are local in the sense that they are not contained in the algebra $\mathscr{A}(O_{\mathscr{R}_2})$ and vice versa.

In the example in Section 3 the spaces $\mathcal{M}_R$ are four-dimensional. This is connected with the overlapping properties of **E**. If one wants to recover usual theories with a four-dimensional space, E should have the $4+1$ overlapping property, i.e., if any $4+1$ elements of a subset x of **E** are overlapping, then x should be overlapping. Is this situation transferable to quantum gravity?

The definition of $\mathcal{M}$ fixes only topological properties. We have in addition defined causal and state-dependent topological structures in Section 4 with the help of ideals. How is it possible to define metrical relations between points of $\mathcal{M}$? One idea is that perhaps correlations between measurements will help to do this.

We hope that the presented method of defining a theory by a base of a net (E, $\preccurlyeq$) will be confirmed by more examples and that the questions above will find answers.

ACKNOWLEDGMENTS

It is a pleasure to thank K. Fredenhagen and R. Haag for many interesting discussions and a critical reading of the manuscript. I also thank D. Buchholz and R.Tscheuschner for helpful hints and comments.

REFERENCES

Bannier, U. (1987). Allgemein kovariante algebraische Quantenfeldtheorie und Rekonstruktion von Raum-Zeit, Thesis, Hamburg, Germany.

Bannier, U. (1988). *Communications in Mathematical Physics*, **118**, 162.

Buchholz, D. (1974). *Communications in Mathematical Physics*, **36**, 287.

Buchholz, D., D'Antoni, C., and Fredenhagen, K. (1987). *Communications in Mathematical Physics*, **111**, 123.

Comfort, W. W., and Gordon, H. (1961). *Proceedings of the American Mathematical Society*, **12**, 327.

Connes, A. (1982). Classification des facteurs, in *Proceedings of Symposia in Pure Mathematics*, No. 38, Part 2, American Mathematical Society, Providence, Rhode Island.

Dimock, J. (1980). *Communications in Mathematical Physics*, **77**, 219.

Doplicher, S., and Longo, R. (1984). *Inventiones Mathematicae*, **75**, 493.

Fredenhagen, K. (1992). On the general theory of quantized fields, in *Proceedings of the 10th Congress on Mathematical Physics, Leipzig 1991*. K. Schmüdgen, ed., Springer-Verlag, Berlin.

Haag, R. (1992). *Local Quantum Physics; Fields, Particles, Algebras*, Springer-Verlag, Berlin.

Haag, R., and Kastler, D. (1964). *Journal of Mathematical Physics*, **5**, 848.

Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.

Helly, E. (1923). *Jahresbericht der Deutschen Mathematischen Vereinigung*, **32**, 175.

Landau, L. J. (1969). *Communications in Mathematical Physics*, **13**, 246.

Radon, J. (1921). *Mathematische Annalen*, **83**, 113.

Russell, B. (1976). *Human Knowledge, Its Scope and Limits*, George Allen and Unwin, London.